

Anomalous scaling in the Bak-Chen-Tang forest fire model

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We reconsider a model introduced by Bak, Chen, and Tang [Phys. Rev. A **38**, 364 (1988)] as a supposedly self-organized critical model for forest fires. We verify again that the model is not critical in two dimensions, as found also by previous authors. But we find that the model does show anomalous scaling (i.e., is critical in the sense of statistical mechanics) in three and four dimensions. We argue that this is due to the fact that fire fronts in more than two dimensions typically contain large holes which allow for large unburnt clusters to survive such fronts. These clusters then allow the next fire to pass earlier than expected naively. We claim that this is a general feature of noisy coupled relaxation oscillators with locally stable refractory states, and we relate these results to recent claims by A. Johansen [Physica D **78**, 196 (1994)]. [S1063-651X(97)51311-0]

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During the last ten years, the concept of self-organized criticality (SOC), proposed by Bak, Tang, and Wiesenfeld in Ref. [1], has been applied to a large number of phenomena. Although a generally accepted and rigorous definition of SOC still doesn't exist, a number of general features are supposed to hold for any model which shows SOC (see also Ref. [2]). (i) There should be scaling laws which in general will be anomalous for local models (for mean field or random neighbor models the exponents in general will be integer or half-integer, i.e. "normal"). These scaling laws should not be "trivial" (such as $m \sim l^3$ for the mass length connection in three dimensions), though triviality is not always easy to define. (ii) There should be no control parameter which has to be fine tuned. Thus the scaling should be a robust phenomenon, in contrast to standard critical phenomena. Again this criterion is less clear cut than one might wish. (iii) SOC usually shows up in slowly driven systems, when the driving rate tends to zero (an advocatus diaboli who insists that this rate should be considered as a control parameter could thus conclude that SOC doesn't exist at all). Typically, such systems become locally unstable when the stress exerted by the driving exceeds some limit, and react with "avalanches" of activity which are large on microscopic scales, but small on macroscopic ones. (iv) Finally, a common feature of many systems with SOC is that the driving is not controlled as usual through a *force*, but through a *flux* (i.e., an extensive quantity) [3].

Phenomena where these features show up include sand piles [1], earthquakes [4,5], pinned surfaces [6,7], and biological evolution [8]. A last application are forest fires with a small growth rate of trees and an even smaller rate for spontaneous ignition ("lightning") [9]. The latter example is special in the sense that it requires *three* different time scales for criticality.

In Ref. [10] Bak, Chen, and Tang (BCT) claimed that this list should also include forest fires without lightning. The specific model studied by BCT used a regular d -dimensional lattice and discrete time, with each lattice site in one of three possible states: green tree, burning tree, or ash. During one time unit, a burning tree ignites all green neighbors (if there are any) and turns itself into ash. This is the fast part of the

dynamics. The slow part is the regrowth of trees. It is modeled by a stochastic spontaneous transition ash \rightarrow tree with probability $p \ll 1$. Thus in each time step a randomly selected fraction p of all ash sites is flipped into trees. As pointed out in Ref. [11], a much better interpretation of the BCT model would be in terms of epidemics with slow recovery resp. slow loss of immunization, but we shall continue to speak about forest fires for convenience.

Simulations [11] on large lattices (up to 4800^2 sites) and for very small values of p (down to 5×10^{-4}) showed, however, that dynamics in $d=2$ is quite different from that suggested by BCT on the basis of small-scale simulations. Instead of being critical, the system develops noisy spiral patterns which become less and less noisy as $p \rightarrow 0$. More precisely, spiral arms (fire fronts) propagate with finite mean velocity v for any p , and the typical distance ξ between spiral arms (the characteristic length scale) as well as the time T between two passages of a front scale as $1/p$: $\xi \propto T \propto 1/p$. In the limit $p \rightarrow 0$ the coherence length thus diverges, implying that the dynamics is governed by *average* tree densities over larger and larger regions. In this limit the dynamics is thus similar to that of coupled relaxation oscillators with very sudden discharge and very slow recharging. This explains qualitatively the patterns found, though any details are still badly understood due to the inherent noise for any $p \neq 0$ which leads to permanently ongoing pattern rearrangements.

Indeed, the authors of Ref. [11] were rather careful in the interpretation of their simulations. They pointed out that if the model were nontrivially critical with characteristic length and time scales scaling as

$$\xi \propto p^{-\alpha}, \quad T \propto p^{-\beta} \quad \alpha, \beta \neq 1, \quad (1)$$

then this could only be the case if fire fronts get fuzzy for $p \rightarrow 0$, and front velocities would tend to zero. In such a case the fire would be at a critical *percolation* threshold (a similar scenario does hold indeed for some versions of forest fire models with lightning [12,13]). The medium into which it percolates would not be uncorrelated, in contrast to standard percolation. Using nevertheless estimates of critical expo-

nents from the latter, it was concluded in [11] that such an alternative scenario was very unlikely but hard to exclude rigorously. The same conclusion (with less caveats in spite of less statistics) was also reached in Ref. [14].

Simulations in three and four dimensions were much less significant. This was partly due to difficulties in visualizing such systems, partly because it is hard to keep a fire from getting extinct for small values of p . Nevertheless some indications for well defined fire fronts were seen in $d=3$ [11], and it was concluded that basically the same scenario holds there as in $d=2$. Obviously, this cannot be true for arbitrarily high dimension. At least for $d \geq 6$ the behavior should be that of dynamic percolation, with small fluctuations of the density of trees around a mean field value. The characteristic time of these oscillations is not $T \sim 1/p$ as for fronts, but $T \sim 1/\sqrt{p}$ [11]. Simulations showed the actual behavior for $d=3$ and $d=4$ was in between both [11,14]. Unfortunately, this was not followed up, and the possibility $T \sim 1/p^\beta$ with $1/2 < \beta < 1$ was not considered seriously.

In a recent paper, Johansen [15] claimed exactly that. In addition, he claimed that the same is true also in $d=2$. On the other hand he confirmed that spirals are formed in $d=2$, and that the typical distance between fire fronts scales as $L \sim 1/p$. Now it is easy to see that the latter statements are self-contradictory. They would imply that fronts propagate with speed $v = L/T \sim p^{-(1-\beta)} \rightarrow \infty$ for $p \rightarrow 0$. Since the front can propagate at most one lattice site in each time step, this is impossible.

In order to clarify this situation, we report in the present letter on simulations where we measured T with high precision, for $d=2, 3$, and 4 , and for wide ranges of p . For $d=3$ we find indeed a very clear indication of anomalous scaling, $\beta = 0.77 \pm 0.02$. The situation is slightly less clear in $d=4$ for reasons detailed below, but we again find scaling (with $\beta \approx 0.6 \pm 0.05$). These findings imply that $pT \rightarrow 0$ for $p \rightarrow 0$. For $d=2$, finally, we find that pT also decreases slightly with decreasing p , but not as a power law. Our data are not precise enough to distinguish clearly between a logarithmic increase,

$$pT \sim [1/\ln(1/p)]^\gamma, \quad \gamma > 0, \quad (2)$$

and a limited increase which leads to a finite value at $p \rightarrow 0$. They favor the latter. But if Eq. (2) would hold, instead, we would have the alternative scenario mentioned above in which fronts become fuzzy, front velocities become zero, and the evolution is basically a critical (correlated) percolation phenomenon.

The latter seems to apply in equal to or more than three dimensions, although the situation is not clear there either. The problem is the following: if $pT \rightarrow 0$ for $p \rightarrow 0$, then the fraction of replenished trees between two peaks of the local fire intensity has to go to zero also. Except for transients this means that also the fraction of trees burnt during such a peak must tend to zero. In such a case we would naively expect that the amplitude of any (noisily) periodic observable should diminish when p is decreased. This is not observed. Instead, the peaks stand out *very* clearly, even for the smallest values of p . Thus our data suggest at first sight that fires burn large areas completely, also in three and four dimensions (they do so in $d=2$, but there it is expected). This

might indicate that even with p values as small as 10^{-3} we are not yet in the asymptotic scaling region, but an alternative scenario will be discussed below.

For our simulations we used the basic algorithm described in Ref. [11]. The current state of the system is encoded in two data structures: a list of burning tree sites, and a bit map indicating for each site whether it contains a tree or not. Notice that we do not have to distinguish in the latter between burning and nonburning trees since that information is contained in the list. Thus we can use bit coding in order to simulate very large lattices. Instead of using d coordinates, each site is indexed by a single integer, and boundary conditions are helical. In each time step, a new list of burning trees is established by scanning through all neighbors of all entries in the old list, and the old list is thereafter replaced by the new one. After this, pN_{ash} sites (N_{ash} is the total number of sites containing ash) are randomly selected and switched from ash to tree.

To avoid that the fire dies out, we used correlated random initial conditions, and we discarded transients which involved at least 100 oscillation periods. If the fire died out nevertheless, we ignited some fires ‘‘by hand,’’ and started the run again.

Lattice sizes went up to 16384×16384 in two dimensions (for $p = 0.00005$), and to comparable numbers of sites in $d \geq 3$. Simulation times went up to $t = 4 \times 10^6$, and were always larger than $100/p$.

Since the aim of the present paper is to obtain precise estimates of T , the biggest effort was devoted to that. As in previous analyses, we used the number of burning trees as observable. But in order to improve the signal to noise ratio, we did *not* simply measure the total number, averaged over the entire lattice. The reason is that fires in different parts of a large lattice will in general not oscillate in phase, whence cross correlations from distant regions will mainly contribute to the noise in the autocorrelation function. We therefore proceeded as follows: we divided the lattice into hypercubes of linear size l with $l < L$, and measured the numbers $n_i(t)$ of burning trees in the i th cube at time t . From each of these local time series we estimated autocovariances

$$c_i(t) = \langle n_i(\tau) n_i(t + \tau) \rangle \quad (3)$$

which were then averaged over the lattice,

$$c(t) = \sum_i c_i(t). \quad (4)$$

Oscillation periods T were either estimated from peak-to-peak distances in $c(t)$ or by Fourier transforming $c(t)$, obtaining thereby maximum entropy spectrum estimate $S(f)$ [16]. Results were the same. Two typical plots of $c(t)$ are shown in Fig. 1.

The sharpness of the peaks in $c(t)$ and the strong higher harmonics result from the fact that $l \ll \xi$, thus $n_i(t)$ is non-zero only during the short time when a fire front passes through cube i . As a consequence we obtain very clean signals and very precise estimates of T .

We should point out here that T is not the only time scale characterizing $c(t)$. First of all, there is also the coherence time τ_{coh} . It can be measured from the asymptotic exponential decay of the oscillation amplitudes. As expected, it

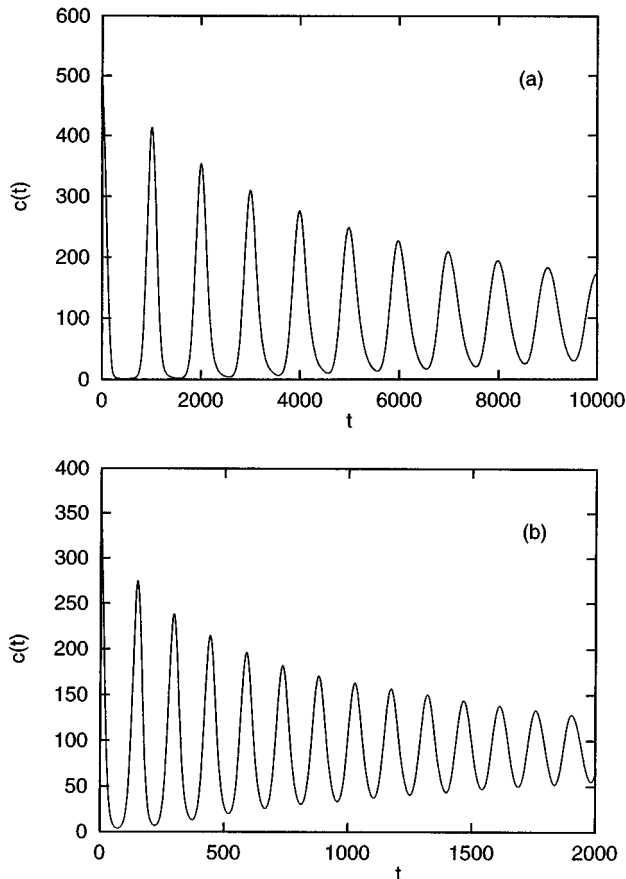


FIG. 1. Autocovariances $c(t)$ with (a) $d=2$, $p=0.001$, lattice size 4096×4096 , 10^6 iterations, and $l=128$; (b) $d=3$, $p=0.003$, lattice size 256^3 , 10^5 iterations, and $l=16$. In both cases, only 16 squares resp. cubes were used in the averaging. Normalization is arbitrary.

grows quickly with $1/p$. But we did not make systematic measurements since the exponential decay is not observed at finite times due to a third time scale, namely the regeneration time $\tau_{\text{regen}}=1/p$. It is easy to see that the amplitudes of all peaks at $t>0$ have to decrease for fixed l and $p \rightarrow 0$, if $T \ll \tau_{\text{regen}}$ in this limit. Indeed, for $l=1$ one finds

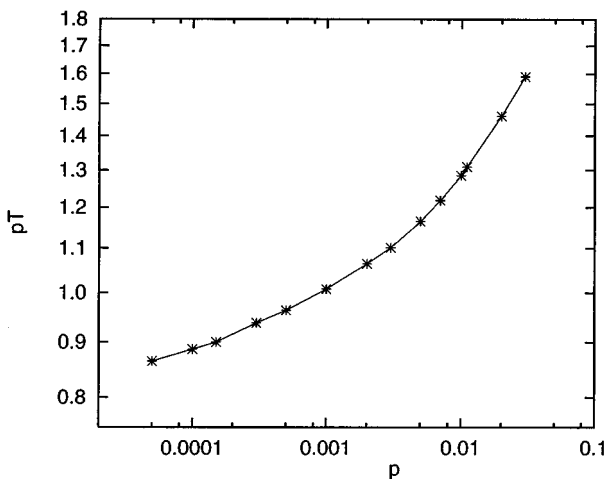


FIG. 2. Log-log plot of pT versus p for two dimensions. Here, T is the average peak-to-peak distance in $c(t)$.

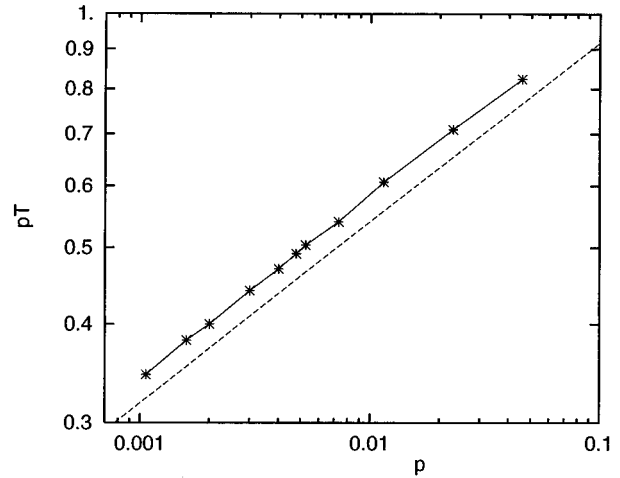


FIG. 3. Same as Fig.2, but for $d=3$. The dashed line corresponds to a power law $T \sim p^{-\beta}$ with $\beta=0.77$.

$c(T)/c(0) \approx pT$. Thus the shape of $c(t)$ depends strongly on the box size l , but we verified that the locations of the maxima (which determine T) are independent of it.

For small values of p (requiring large systems and long simulation times) it was not feasible to store all $n_i(t)$ for each cube i and every t . In such cases we decimated the data by either reading out only a fraction of cubes, or by coarse graining in t and storing $n'_i(kt) = \sum_{\tau=kt}^{(k+1)t} n_i(\tau)$ for some integer $k > 1$. Both gave nearly the same performance as without decimation, even if the data were reduced by more than one order of magnitude.

Our final results are shown in Figs. 2–4. Each of them is a log-log plot showing pT versus p . In Figs. 2 and 3 the estimated errors are smaller than the symbols. We see clearly the trends discussed above. For $d=4$ (Fig. 4) the errors are larger and not purely statistical: runs with different initial conditions gave occasionally values which differed by more than naive statistical error estimates. Obviously this means that either the system is not ergodic, or that our simulation times were not sufficient to explore all phase space. Nevertheless we believe that the data give a clear indication of scaling also for $d=4$.

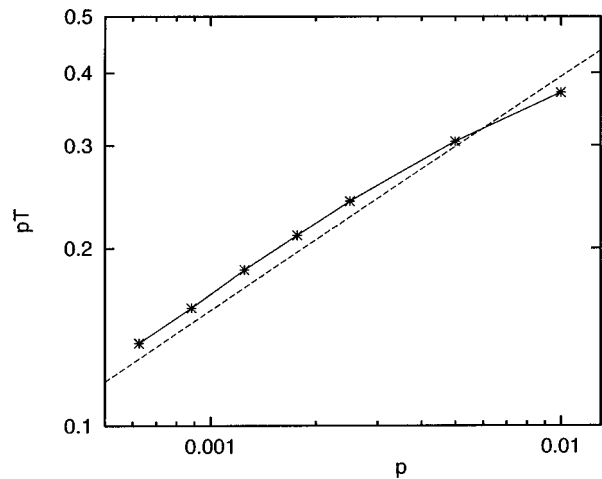


FIG. 4. Same as Fig. 2, but for $d=4$. The dashed line corresponds to $\beta=0.6$.

As pointed out above, the most straightforward interpretation of our data for $d \geq 3$ is that there are fronts with distances $\xi \approx vT \sim p^{-\alpha}$. Between two successive fronts only a tiny fraction of trees can regrow, and also a tiny fraction of sites could burn when the front passes. If these fronts would propagate into an essentially unstructured medium, this would imply that the process is close to critical percolation, and it would be hard to understand the very large and regular amplitudes seen, e.g., in Fig. 1(b). To understand better what is going on we tried to visualize typical three-dimensional configurations, but only with moderate success. But the above suggests that fronts do not propagate into an unstructured medium. It is well known that the complement of a slightly supercritical percolation cluster in $d \geq 3$ is connected. Thus a passing fire, even if it is supercritical and has therefore a sharp front, could still leave intact connected regions with slightly subcritical or even supercritical tree densities. This would allow the next front to pass very soon again. The crucial point is that this front should propagate only on a sparse but connected subset of trees, requiring the patterns of trees to be highly structured.

In summary, we have shown that there are anomalous scaling laws in the Bak-Chen-Tang forest fire models, but only in higher dimensions. This suggests that there are sharp fire fronts in $d > 2$, even in the limit $p \rightarrow 0$, but each front propagates only on a sparse subset of trees. In this way the fire can be endemic in $d \geq 3$, burning only an infinitesimal

fraction of trees between two recurrences to the same mesoscopic region without, nevertheless, resembling critical percolation.

Since the present model can be considered as a model for extremely noisy coupled relaxation oscillators, it is an interesting question whether this is the generic behavior of noisy coupled relaxation oscillators in more than two dimensions.

Another final remark concerns the relationship of the present model with other SOC models. The main difference is that in the present model the system is slowly driven (by the growth of trees) into a *susceptible* state, while most other SOC models are driven into *unstable* states which will “topple” (discharge, catch fire, etc.) spontaneously. This difference would be blurred if a susceptible site has a small chance to topple anyhow, as in the Drossel-Schwabl [9] model. But it is also blurred if the connectivity of the lattice is so high that activity can efficiently spread over large distances without leaving many traces. This is obviously what happens in the present model when $d \geq 3$. This explains why the model shows SOC for $d \geq 3$, but not for $d = 2$.

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